2021年3月17日

§9.4 Coxeter Groupoid

Recall: P44 W is called Coxeter group, $S \subseteq W$, element order 2
if $\langle S \mid (ss')^{m(s,s')} = 1$ for all $s, s' \in S$ with $m(s,s') < \infty \rangle$
$\xrightarrow{\cong} W$, $s \mapsto s$ is a group isomorphism

Cor 9.2.22: Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan group
Let $X \in \mathcal{X}$ and $i, j \in I$ with $i \neq j$.
If m_{ij}^X is finite then
(1) $id_X (s_i s_j)^{m_{ij}^X} = id_X$
(2) $\mathrm{Prod}_{ij}^X (m_{ij}^X) = \mathrm{Prod}_{ji}^X (m_{ij}^X)$
↳ Coxeter relation
$\mathrm{Prod}_{ij}^X (2k) = id_X (s_i s_j)^k$, $\mathrm{Prod}_{ij}^X (2k+1) = id_X (s_i s_j)^k s_i$

Defn. Let X be a set and Let G ... directed graph,
with $\mathcal{X}$ as its set of vertices.
~~~ is an $\mathcal{X}$-graph.
The free category generated by G is the category with $\mathcal{X}$ as the set of objects, where the morphisms are admissible finite compositions of arrows of G

For a category C and any two objects X, Y of C,
Let $R_{X,Y} \subseteq \mathrm{Hom}(X,Y) \times \mathrm{Hom}(X,Y)$ be a relation
that is, a subset.
(relation: refexive, symmetric, and transive)
Then there exists a category C/R and a functor $F_R: C \to C/R$ with the following properties.
(1) If $(f, f') \in R_{X,Y}$, then $F_R(f) = F_R(f')$
(2) Let $\mathcal{D}$ be a category and $H: C \to \mathcal{D}$ a functor
If $H(f) = H(f')$, $\forall f, f' \in \mathrm{Hom}(X,Y)$, $X, Y \in C$ with $(f, f') \in R_{X,Y}$
then there exists a unique functor
$H': C/R \to D$, s.t. $H' F_R = H$

$$\begin{tikzcd} C \arrow[r, "F_R"] \arrow[dr, "H"'] & C/R \arrow[d, dashed, "\exists H'"] \\ & D \end{tikzcd}$$

Pf: (1) Suppose $\sigma \in \mathrm{Hom}(X,Y)$, $F: C \to C/R$
$\sigma \mapsto [\sigma]$
$X \mapsto X$
$F_R(f) = [\sigma] = F_R(f')$
(2) $H(X) = X'$, $H(Y) = Y'$, $X', Y' \in D$
$H'(F_R(f)) = H(f)$, $H'(X) = X'$

The functor F_R is then necessarity a bijection between the objects of C and the objects of C/R
If C is ... the free category generated by a graph G,
then C/R is called the category with generators G and relations R.

Defn 9.4.1, Let I ... non-empty finite set, $\mathcal{X}$ non-empty set.
G a directed labeled graph with $\mathcal{X}$ ... objects,
s.t. each object has 1 ... incoming and outgoing arrow labeled by i
let $r_i(X)$ be the target of the i-arrow starting at X. 保证 $r_i(X)$ 是一个置换

$X \underset{i}{\overset{i}{\rightleftarrows}} r_i(X)$

For all $X \in \mathcal{X}$, let $M^X = (m_{ij}^X)_{i,j \in I} \in (\mathbb{N} \cup \{\infty\})^{I \times I}$
be a symmetric matrix, s.t. $m_{ii}^X = 1$ for all $i \in I$
Assume that $(r_i r_j)^{m_{ij}^X}(X) = X$ for $\forall X \in \mathcal{X}$
...... $m_{ij}^X \neq \infty$
The Coxeter groupoid $\mathrm{Cox}(G, (M^X)_{X \in \mathcal{X}})$ is the category with generators G and relations
$id_X (s_i s_j)^{m_{ij}^X} = id_X$
s_i^X -- morphism corresponding to the i-arrow of G starting at X.

$i \xrightarrow{r_i} j$

$m_{ii}^X = 1 \Rightarrow s_i^{r_i(X)} s_i^X = id_X$ $r_i(r_i(X)) = X$ r_i

Example 9.4.2. Let $I = \{1, \ldots, n\}$ let G -- directed graph with one vertex with one loop for each $i \in I$,
Let $M = (m_{ij})_{i,j \in I} \in (\mathbb{N} \cup \{\infty\})^{n \times n}$ be a symmetric matrix with $m_{ii} = 1$ for all $i \in I$
Then $\mathrm{Cox}(G, M)$ is a Coxeter group viewed as a category

Defn. 9.4.3 Let $\mathrm{Cox}(G, (M^X)_{X \in \mathcal{X}})$ be a Coxeter groupoid.
$\mathcal{X}$ a set, G $\mathcal{X}$-graph.
$\forall X, Y \in \mathcal{X}$ and $w \in \mathrm{Hom}(Y, X)$
Let $l(w)$ be the smallest integer $k \geq 0$
s.t. $w = id_X s_{i_1} \cdots s_{i_k}$, for some $i_1 \cdots i_k \in I$,
The family $(l: \mathrm{Hom}(X,Y) \to \mathbb{N}_0)_{X, Y \in \mathcal{X}}$
is called the length function

$l(w) = k$

$l(w) = k$

Lemma 9.4.4 Let $\mathrm{Cox}(G, (M^X)_{X \in \mathcal{X}})$ be a Coxeter groupoid
There is a unique functor
$\det: \mathrm{Cox}(G, (M^X)_{X \in \mathcal{X}}) \to \mathcal{D}(\mathcal{X}, \{-1, 1\})$
which is the identity on $\mathcal{X}$
sends any $s_i^X \in \mathrm{Hom}(X, r_i(X))$ to $(r_i(X), -1, X)$

Pf:

$$\begin{tikzcd} D(G) \arrow[r] \arrow[dr] & \mathrm{Cox}(G) \arrow[d, dashed, "\exists \det"] \\ & D(\mathcal{X}, \{-1, 1\}) \end{tikzcd}$$

id_X — 偶数个 $s_i s_j$
$-1 \cdot -1 = 1$
3三性质

Lem 9.4.5 Let $\mathrm{Cox}(G, (M^X)_{X \in \mathcal{X}})$

Let $X, Y, Z \in \mathcal{X}$ and let $w: X \to Y$, $w': Y \to Z$
be morphisms in $\mathrm{Cox}(G, (M^X)_{X \in \mathcal{X}})$, $k \geq 0$
and $i_1, \ldots, i_k \in I$ then
(1) $|l(w) - l(w')| \leq l(w'w) \leq l(w') + l(w)$,
$l(w^{-1}) = l(w)$
(2) $l(w'w) \equiv l(w') + l(w) \mod 2$
(3) $l(s_i w), l(w s_i) \in \{l(w)+1, l(w)-1\}$ for all $i \in I$
(4) $k - l(id_X s_{i_1} \cdots s_{i_k})$ is a non-negative even integer

Pf: P319 / Lem 9.1.13, Lemma 9.4.4.
(1) by Defn of l, $l(w^{-1}) = l(w)$,
$l(w'w) \leq l(w) + l(w')$
$\therefore l(w) \leq l(w'^{-1}) + l(w'w)$, $\therefore l(w) - l(w') \leq l(w'w)$
$l(w') - l(w) \leq l(w'w)$
(2) and (4) follow from Lem. 9.4.4
(3) from (1) and (2)

Defn. 9.4.6, Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph.
Let G be the $\mathcal{X}$-graph with arrows labeled by the elements of I.
s.t. for any $i \in I$ and $X \in \mathcal{X}$,
there is precisely one i-arrow starting at X
... $r_i(X)$
$\forall X \in \mathcal{X}$ Let $M^X = (m_{ij}^X)_{i,j \in I}$

m_{ij}^X $r_i r_j (\;)^{m_{ij}^X} = 1$

$N_0(\alpha_j)$
★ M_{ij}^X P_{320}
$(s_i s_j)^{m_{ij}}$

$\mathrm{Cox}(G) = \mathrm{Cox}(G, (M^X)_{X \in \mathcal{X}})$ is the Coxeter groupoid of G.
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