

§ 9.4 Coxeter Groupoid

Recall: Part W is called Coxeter group, $S \subseteq W$, element order if $\langle s, s' \rangle^{m(s,s')} = 1$ for all $s, s' \in S$ with $m(s, s') < \infty$

Cor 9.2.22: Let $G = G(I, X, r, A)$ be a Cartan group
 Let $X \in X$ and $i, j \in I$ with $i \neq j$,
 If m_{ij}^X is finite then

- (1) $id_X(s_i s_j)^{m_{ij}^X} = id_X$
 - (2) $Prod_{ij}^X(m_{ij}^X) = Prod_{ji}^X(m_{ij}^X)$
- \hookrightarrow Coxeter relation
 $Prod_{ij}^X(2k) = id_X(s_i s_j)^k$, $Prod_{ij}^X(2k+1) = id_X(s_i s_j)^k s_i$

Defn. Let X be a set and Let $G \dots$ directed graph, with X as its set of vertices.
 \dots is an X -graph.

The free category generated by G is the category with X as the sets of objects, where the morphisms are admissible finite compositions of arrows of G

For a category C and any two objects X, Y of C ,
 Let $R_{X,Y} \subseteq Hom(X, Y) \times Hom(X, Y)$ be a relation that is, a subset.

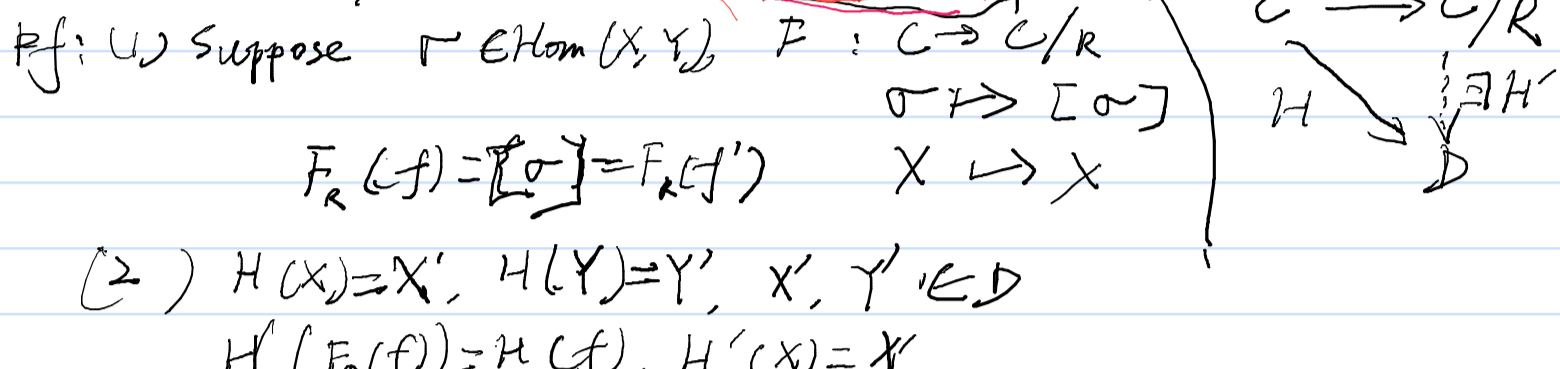
(relation: reflexive, symmetric, and transitive)

Then there exists a category C/R and a functor $F_R: C \rightarrow C/R$ with the following properties.

- (1) If $(f, f') \in R_{X,Y}$, then $F_R(f) = F_R(f')$
- (2) Let \mathcal{D} be a category and $H: C \rightarrow \mathcal{D}$ a functor
 If $H(f) = H(f')$, $\forall f, f' \in Hom(X, Y)$, $X, Y \in C$ with $(f, f') \in R_{X,Y}$

then there exists a unique functor

$H': C/R \rightarrow \mathcal{D}$, s.t. $H' \circ F_R = H$



- (2) $H(X) = X'$, $H(Y) = Y'$, $X', Y' \in \mathcal{D}$
 $H'(F_R(f)) = H(f)$, $H'(X) = X'$

The functor F_R is then necessarily a bijection between the objects of C and the objects of C/R

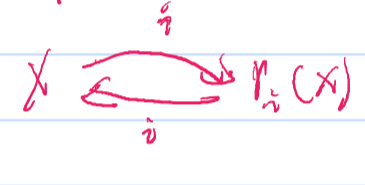
If C is the free category generated by a graph G ,

then C/R is called the category with generators G and relations R .

Defn 9.4.1, Let $I \dots$ non-empty finite set, X non-empty set.
 G a directed labeled graph with $X \dots$ objects, s.t. each object has \dots incoming and outgoing arrow labeled by i .

Let $r_i(X)$ be the target of the i -arrow starting at X . *保证 $r_i(X)$ 是一个置换*

For all $X \in X$, let $M^X = (m_{ij}^X)_{i,j \in I} \in (N \cup \{\infty\})^{I \times I}$
 be a symmetric matrix, s.t. $m_{ii}^X = 1$ for all $i \in I$



Assume that $(r_i r_j)^{m_{ij}^X}(X) = X$ for $\forall X \in X$
 $\dots, m_{ij}^X \neq \infty$

The Coxeter groupoid $Cox(G, (M^X)_{X \in X})$ is the category with generators G and relations

$id_X(s_i s_j)^{m_{ij}^X} = id_X$

$S_i^X \dots$ morphism corresponding to the i -arrow of G starting at X .

$m_{ij}^X = 1 \Rightarrow S_i^{r_i(X)} S_j^X = id_X$ $r_i(r_i(X)) = X$

Example 9.4.2. Let $I = \{1, \dots, n\}$, let $G \dots$ directed graph with one vertex with one loop for each $i \in I$,
 let $M = (m_{ij})_{i,j \in I} \in (N \cup \{\infty\})^{n \times n}$ be a symmetric matrix with $m_{ii} = 1$ for all $i \in I$

Then $Cox(G, M)$ is a Coxeter group viewed as a category

Defn. 9.4.3 Let $Cox(G, (M^X)_{X \in X})$ be a Coxeter groupoid.
 X a set, G X -graph.

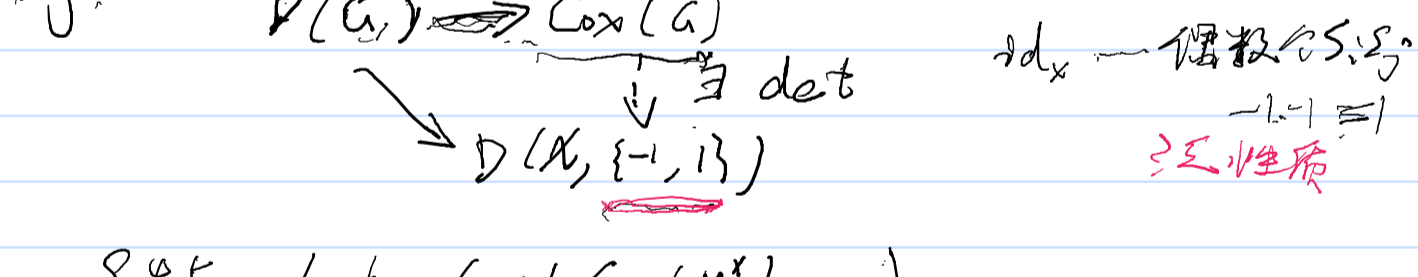
$\forall X, Y \in X$ and $w \in Hom(Y, X)$

Let $l(w)$ be the smallest integer $k \geq 0$ s.t. $w = id_X s_{i_1} \dots s_{i_k}$, for some $i_1, \dots, i_k \in I$

The family $(l: Hom(X, Y) \rightarrow N_0)_{X, Y \in X}$ is called the length function *$l(w) = k$*

Lemma 9.4.4. Let $Cox(G, (M^X)_{X \in X})$ be a Coxeter groupoid

There is a unique functor $det: Cox(G, (M^X)_{X \in X}) \rightarrow \mathcal{D}(X, \{-1, 1\})$ which is the identity on $\dots X$ sends any $S_i^X \in Hom(X, r_i(X))$ to $(r_i(X), -1, X)$



Len 9.4.5 Let $Cox(G, (M^X)_{X \in X})$

Let $X, Y, Z \in X$ and let $w: X \rightarrow Y$, $w': Y \rightarrow Z$ be morphisms in $Cox(G, (M^X)_{X \in X})$, $k \geq 0$ and $i_1, \dots, i_k \in I$ then

- (1) $|l(w) - l(w')| \leq l(w'w) \leq l(w') + l(w)$, $l(w^{-1}) = l(w)$
- (2) $l(w'w) \equiv l(w') + l(w) \pmod{2}$
- (3) $l(s_i w), l(w s_i) \in \{l(w) + 1, l(w) - 1\}$ for all $i \in I$
- (4) $k - l(id_X s_{i_1} \dots s_{i_k})$ is a non-negative even integer

Pf: $P.1.9 / Lem 9.1.3$, Lemma 9.4.4

(1) by Defn of l , $l(w^{-1}) = l(w)$
 $l(w'w) \leq l(w') + l(w)$
 $\therefore l(w) \leq l(w^{-1}) + l(w'w)$, $\therefore l(w) - l(w') \leq l(w'w)$
 $l(w') - l(w) \leq l(w'w)$

(2) and (4) follow from Lem. 9.4.4

(3) from (1) and (2)

Defn. 9.4.6. Let $G = G(I, X, r, A)$ be a Cartan graph. Let G be the X -graph, with arrows labeled by the elements of I .

s.t. for any $i \in I$ and $X \in X$, there is precisely one i -arrow starting at X
 $\dots r_i(X)$

$\forall X \in X$ Let $M^X = (m_{ij}^X)_{i,j \in I}$

$Cox(G) = Cox(G, (M^X)_{X \in X})$ is the Coxeter groupoid of G .

